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STANDARD DEVIATION OF THE GEODETIC QUADRILATERAL POINT COORDINATES DETERMINED BY THE APPLICATION OF HANSEN'S METHOD

STANDARDNA ODSTUPANJA KOORDINATA TAČAKA GEODETSKOG ČETVOROUGLA ODREĐENIH PRIMENOM HANZENOVOG POSTUPKA

Ganić Aleksandar[1], Milutinović Aleksandar[1], Gojković Zoran[1], Mikanović Radenko[2], Vidanović Nebojša[1]

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Abstract: The Danish astronomer Peter Andreas Hansen, who also worked on theoretical geodesy, provided an original solution for solving the geodetic quadrilateral in which the four angles and the length of one side are known. This procedure, identified in scientific literature as Hansen's problem, has been applied in geodesy but significantly more so when solving various problems related to mining measurements in underground exploitation. Nonetheless, the literature does not cover the determining of the standard deviation of the coordinates of unknown points which are established using Hansen's method. In this paper then, equations are derived to calculate the standard deviation of the coordinates determined by the above mentioned procedure.

Key words: geodetic quadrilateral, Hansen's problem, standard deviation

Apstrakt: Danski astronom Peter Andreas Hansen, koji se bavio i teorijskom geodezijom dao je originalno rešenje za rešavanje geodetskog četvorougla u kome su poznata četiri ugla i dužina jedne strane. Ovaj postupak koji se u stručnoj literaturi naziva "Hanzenov problem" našao je primenu u Geodeziji, ali znatno više u rešavanju različitih zadataka iz oblasti Rudarskih merenja koji se javljaju pri podzemnoj eksploataciji. Međutim, ono što u literaturi nije obrađeno predstavlja određivanje standardnih odstupanja koordinata nepoznatih tačaka do kojih se dolazi primenom Hanzenovog postupka. U ovom radu izvedene su jednačine za računanje standardnih odstupanja koordinata određenih navedenim postupkom.

Ključne reči: geodetski četvorougao, Hanzenov problem, standardno odstupanje

[1] University of Belgrade – Faculty of Minig and Geology, Đušina 7, 11000 Belgrade, Serbia,
e-mails: aleksandar.ganic@rgf.bg.ac.rs; aleksandar.milutinovic@rgf.bg.ac.rs; zoran.gojkovic@rgf.bg.ac.rs; nebojsa.vidanovic@rgf.bg.ac.rs
[2] University of Banja Luka – Faculty of Mine Engineering, Save Kovačevića bb, 79101 Prijedor, Republic of Srpska, Bosnia and Herzegovina; e-mail: radenko.mikanovic@unibl.rs

1. INTRODUCTION

Peter Andreas Hansen (1795- 1874) was a Danish astronomer whose most famous work is in the field of celestial mechanics: theories on the motion of comets, small planets, the Moon, as well as lunar tables (Ephemerides). In addition to astronomy, Hansen worked on optics, probability theory and theoretical geodesy. In the field of Geodesy, he is well-known for his method of solving the geodetic quadrilateral in which four angles and the length of one side are known. Based on these elements, the unknown angles φ and ψ (Figure 1) are calculated, followed by all the other elements of the quadrilateral.

Such a way of solving the geodetic quadrilateral, known as Hansen's problem can be applied in:

- Determining the elements of eccentricity (Mihailović, 1987);
- Determining the base for terrestrial photogrammetric surveying (Schofield and Breach, 2007);
- Determining the points for vertical shaft surveys (Patarić, 1990);
- Connecting pits to the base on the surface of the terrain (Baturić, 1959), etc.

On the whole, Hansen's problem is applied in the instances where two points of the quadrilateral are inaccessible or are, for reasons of safety, inconvenient for stations at which it is necessary to focus the surveying instrument. The four angles required in the quadrilateral are measured only from the remaining two points of the quadrilateral.

When it is necessary to evaluate the coordinates of the points in the geodetic quadrilateral, then it is, as in every free trigonometrical network, essential to know the coordinates of the two points. In that case, the length of the side between these two points is calculated on the basis of the coordinates of the given points, while only angles are measured on the terrain. It follows that two situations are then feasible:

- The angles are measured from the given points, and
- The angles are measured from unknown points.

The first situation, where the angles are measured from the given points is more straightforward and does not involve calculating the elements of the geodetic quadrilateral. Based on the directional angle of the side between the given points and the measured angles, oriented directions are calculated in terms of unknown points or their coordinates (Chandra, 2005).

The second situation, when the angles are measured from unknown points, is more complex and requires that the geodetic quadrilateral be previously solved in accordance with Hansen's method.

The five known elements (four angles and one side) represent the mathematical minimum hence providing a unique solution. All additional measurements of an element of the geodetic quadrilateral (redundant measurements) offer an ambiguous solution necessitating the leveling of all measured values. In this case the calculations for the coordinates of unknown angles will be carried out by applying the methods of indirect leveling, thus obtaining the standard deviation of the coordinates of the unknown points.

2. CALCULATING ELEMENTS OF THE GEODETIC QUADRILATERAL USING HANSEN'S METHOD

The geodetic quadrilateral problem which consists of two given but inaccessible points and two unknown but accessible points can be solved using Hansen's method (www2.washjeff).

Points *A* and *B* are given points with known coordinates Y_A and X_A, or Y_B and X_B, but they are inaccessible and angles cannot be measured from them. The horizontal length *d* between them is also known and is calculated on the basis of the coordinates of these points.

In order to solve the quadrilateral or to determine the coordinates of the unknown but accessible points *1* and *2* it is sufficient to measure, on the terrain, the horizontal angles from these points: α_1, α_2, β_1, β_2.

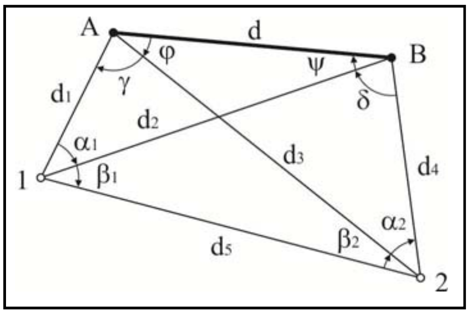

Figure 1 - Elements of the geodetic quadrilateral which is solved using Hansen's method

On account of measured and known (given) quantities, all unknown quantities in the quadrilateral are calculated: angles γ, δ, φ, ψ and lengths d_1, d_2, d_3, d_4 and d_5.

The unknown angles γ and δ are derived from triangles:

from ΔA12: $$\gamma = 180^\circ - \alpha_1 - \beta_1 - \beta_2 \tag{1}$$

from ΔB12: $$\delta = 180^\circ - \alpha_2 - \beta_1 - \beta_2 \tag{2}$$

The unknown angles φ and ψ are calculated on the basis of their semi-sums and semi-differences, or:

$$\varphi = \frac{\varphi + \psi}{2} + \frac{\varphi - \psi}{2} \tag{3}$$

$$\psi = \frac{\varphi + \psi}{2} - \frac{\varphi - \psi}{2} \tag{4}$$

Given that $\varphi + \psi = \beta_1 + \beta_2$ (from ΔAB1 and ΔA12), the semi-sum of the unknown angles is:

$$\frac{\varphi + \psi}{2} = \frac{\beta_1 + \beta_2}{2} \tag{5}$$

In accordance with the Sine rule:

$$\frac{d}{d_1}=\frac{\sin\alpha_1}{\sin\psi};\quad \frac{d_1}{d_5}=\frac{\sin\beta_2}{\sin\gamma};\quad \frac{d_5}{d_4}=\frac{\sin\delta}{\sin\beta_1};\quad \frac{d_4}{d}=\frac{\sin\varphi}{\sin\alpha_2} \tag{6}$$

After multiplying:

$$\frac{d}{d_1}\cdot\frac{d_1}{d_5}\cdot\frac{d_5}{d_4}\cdot\frac{d_4}{d}=1=\frac{\sin\alpha_1\cdot\sin\beta_2\cdot\sin\delta\cdot\sin\varphi}{\sin\psi\cdot\sin\gamma\cdot\sin\beta_1\cdot\sin\alpha_2} \tag{7}$$

Or:

$$\tan\mu=\frac{\sin\varphi}{\sin\psi}=\frac{\sin\gamma\cdot\sin\beta_1\cdot\sin\alpha_2}{\sin\alpha_1\cdot\sin\beta_2\cdot\sin\delta} \tag{8}$$

As:

$$\frac{\tan\frac{\varphi-\psi}{2}}{\tan\frac{\varphi+\psi}{2}}=\frac{\tan\mu-1}{1+\tan\mu}=\tan\left(\mu-45^\circ\right) \tag{9}$$

The semi-difference of the unknown angles is:

$$\frac{\varphi-\psi}{2}=\arctan\left[\tan\frac{\varphi+\psi}{2}\cdot\tan\left(\mu-45^\circ\right)\right] \tag{10}$$

The unknown sides of the quadrilateral d_1 and d_4, as well as its diagonal d_2 and d_3, are calculated using the Sine rule based on the known side d:

$$\begin{aligned} d_1&=\frac{\sin\psi}{\sin\alpha_1}\cdot d;\quad d_2=\frac{\sin(\gamma+\varphi)}{\sin\alpha_1}\cdot d;\\ d_3&=\frac{\sin(\delta+\psi)}{\sin\alpha_2}\cdot d;\quad d_4=\frac{\sin\varphi}{\sin\alpha_2}\cdot d \end{aligned} \tag{11}$$

While side d_5 is calculated in terms of the previously evaluated sides (11):

$$d_5=\frac{\sin\gamma}{\sin\beta_2}\cdot d_1=\frac{\sin\delta}{\sin(\alpha_2+\beta_2)}\cdot d_2=\frac{\sin\gamma}{\sin(\alpha_1+\beta_1)}\cdot d_3=\frac{\sin\delta}{\sin\beta_1}\cdot d_4 \tag{12}$$

The coordinates of the unknown points *1* and *2* are calculated on the basis of the evaluated elements of the quadrilateral, in terms of the known points *A* and *B,* using the equations:

$$\begin{aligned} Y_1&=Y_A+d_1\cdot\sin\nu_A^1=Y_A+d_1\cdot\sin\left(\nu_A^B+\varphi+\gamma\right)\\ X_1&=X_A+d_1\cdot\cos\nu_A^1=X_A+d_1\cdot\cos\left(\nu_A^B+\varphi+\gamma\right) \end{aligned} \tag{13}$$

Or

$$\begin{aligned} Y_1&=Y_B+d_2\cdot\sin\nu_B^1=Y_B+d_2\cdot\sin\left(\nu_B^A-\psi\right)\\ X_1&=X_B+d_2\cdot\cos\nu_B^1=X_B+d_2\cdot\cos\left(\nu_B^A-\psi\right) \end{aligned} \tag{14}$$

As:

$$\begin{aligned} Y_2&=Y_B+d_4\cdot\sin\nu_B^2=Y_B+d_4\cdot\sin\left(\nu_B^A-\psi-\delta\right)\\ X_2&=X_B+d_4\cdot\cos\nu_B^2=X_B+d_4\cdot\cos\left(\nu_B^A-\psi-\delta\right) \end{aligned} \tag{15}$$

Or

$$
\begin{aligned}
Y_2 &= Y_A + d_3 \cdot \sin \nu_A^2 = Y_A + d_3 \cdot \sin\left(\nu_A^B + \varphi\right) \\
X_2 &= X_A + d_3 \cdot \cos \nu_A^2 = X_A + d_3 \cdot \cos\left(\nu_A^B + \varphi\right)
\end{aligned}
\tag{16}
$$

3. STANDARD DEVIATION OF THE COORDINATES OF UNKNOWN POINTS

The standard deviation of the coordinates of unknown points *1* and *2,* due to extensive deriving, will be evaluated only on the basis of equations (13) and (15).

Previously, in these equations, all evaluated values will be expressed only in terms of measured angles α_1, α_2, β_1 and β_2:

$$Y_1 = Y_A + \sin\left[\nu_A^B + (B+A) + \left(180^\circ - \alpha_1 - \beta_1 - \beta_2\right)\right] \cdot \frac{\sin(B-A)}{\sin\alpha_1} \cdot d \tag{17}$$

$$X_1 = X_A + \cos\left[\nu_A^B + (B+A) + \left(180^\circ - \alpha_1 - \beta_1 - \beta_2\right)\right] \cdot \frac{\sin(B-A)}{\sin\alpha_1} \cdot d \tag{18}$$

$$Y_2 = Y_B + \sin\left[\nu_B^A - (B-A) - \left(180^\circ - \alpha_2 - \beta_1 - \beta_2\right)\right] \cdot \frac{\sin(B+A)}{\sin\alpha_2} \cdot d \tag{19}$$

$$X_2 = X_B + \cos\left[\nu_B^A - (B-A) - \left(180^\circ - \alpha_2 - \beta_1 - \beta_2\right)\right] \cdot \frac{\sin(B+A)}{\sin\alpha_2} \cdot d \tag{20}$$

Where shifts are introduced:

$$A = \arctan\left(\frac{\sin(\alpha_1+\beta_1+\beta_2)\cdot\sin\alpha_2\cdot\sin\beta_1 - \sin(\alpha_2+\beta_1+\beta_2)\cdot\sin\alpha_1\cdot\sin\beta_2}{\sin(\alpha_1+\beta_1+\beta_2)\cdot\sin\alpha_2\cdot\sin\beta_1 + \sin(\alpha_2+\beta_1+\beta_2)\cdot\sin\alpha_1\cdot\sin\beta_2}\cdot\tan\frac{\beta_1+\beta_2}{2}\right)$$

$$B = \frac{\beta_1+\beta_2}{2}$$

The standard deviation of the coordinates of unknown points *1* and *2* will be calculated only on the basis of standard deviations of measured angles α_1, α_2, β_1 and β_2, while the coordinates of the given points *A* and *B,* as well as the value of the grid bearing ν_A^B and the length *d* will be considered true values.

In accordance with that, the standard deviation of point coordinates will be calculated using the equations:

$$\sigma_{Y_1} = \sqrt{\left(\frac{\partial Y_1}{\partial\alpha_1}\right)^2\cdot\left(\frac{\sigma_{\alpha_1}}{\rho''}\right)^2 + \left(\frac{\partial Y_1}{\partial\alpha_2}\right)^2\cdot\left(\frac{\sigma_{\alpha_2}}{\rho''}\right)^2 + \left(\frac{\partial Y_1}{\partial\beta_1}\right)^2\cdot\left(\frac{\sigma_{\beta_1}}{\rho''}\right)^2 + \left(\frac{\partial Y_1}{\partial\beta_2}\right)^2\cdot\left(\frac{\sigma_{\beta_2}}{\rho''}\right)^2} \tag{21}$$

$$\sigma_{X_1} = \sqrt{\left(\frac{\partial X_1}{\partial\alpha_1}\right)^2\cdot\left(\frac{\sigma_{\alpha_1}}{\rho''}\right)^2 + \left(\frac{\partial X_1}{\partial\alpha_2}\right)^2\cdot\left(\frac{\sigma_{\alpha_2}}{\rho''}\right)^2 + \left(\frac{\partial X_1}{\partial\beta_1}\right)^2\cdot\left(\frac{\sigma_{\beta_1}}{\rho''}\right)^2 + \left(\frac{\partial X_1}{\partial\beta_2}\right)^2\cdot\left(\frac{\sigma_{\beta_2}}{\rho''}\right)^2} \tag{22}$$

$$\sigma_{Y_2} = \sqrt{\left(\frac{\partial Y_2}{\partial\alpha_1}\right)^2\cdot\left(\frac{\sigma_{\alpha_1}}{\rho''}\right)^2 + \left(\frac{\partial Y_2}{\partial\alpha_2}\right)^2\cdot\left(\frac{\sigma_{\alpha_2}}{\rho''}\right)^2 + \left(\frac{\partial Y_2}{\partial\beta_1}\right)^2\cdot\left(\frac{\sigma_{\beta_1}}{\rho''}\right)^2 + \left(\frac{\partial Y_2}{\partial\beta_2}\right)^2\cdot\left(\frac{\sigma_{\beta_2}}{\rho''}\right)^2} \tag{23}$$

$$\sigma_{X_2}=\sqrt{\left(\frac{\partial X_2}{\partial\alpha_1}\right)^2\cdot\left(\frac{\sigma_{\alpha_1}}{\rho''}\right)^2+\left(\frac{\partial X_2}{\partial\alpha_2}\right)^2\cdot\left(\frac{\sigma_{\alpha_2}}{\rho''}\right)^2+\left(\frac{\partial X_2}{\partial\beta_1}\right)^2\cdot\left(\frac{\sigma_{\beta_1}}{\rho''}\right)^2+\left(\frac{\partial X_2}{\partial\beta_2}\right)^2\cdot\left(\frac{\sigma_{\beta_2}}{\rho''}\right)^2} \tag{24}$$

Where:

$\sigma_{\alpha_1};\sigma_{\alpha_2};\sigma_{\beta_1};\sigma_{\beta_2}$ - standard deviations of measured angles;

and partial derivatives of functions are:

$$\begin{aligned}\frac{\partial Y_1}{\partial\alpha_1}&=d\cdot\csc\alpha_1\Big\{-\cot\alpha_1\cdot\sin\left(\nu_A^B+180^\circ-\alpha_1+A-B\right)\cdot\sin\left(B-A\right)-\\&-\frac{\sin\left(\alpha_2+\beta_2\right)\cdot\sin^2B\cdot\sin\left(B-A\right)\cdot\cos\left(\nu_A^B+180^\circ-\alpha_1+A-B\right)}{C\left(\sec^2B\right)\left(C+2D\cos2B\right)+D^2\left(1+\tan^2B\right)}\cdot\\&\cdot\frac{\left[\sin\left(2\alpha_1-\alpha_2-\beta_2\right)-\sin\left(2\alpha_1-\alpha_2+\beta_2\right)+2\sin\left(\alpha_2+\beta_2\right)+\sin\left(2\alpha_1-\alpha_2+2\beta_1+\beta_2\right)-\sin\left(2\alpha_1+\alpha_2+2\beta_1+\beta_2\right)\right]}{C\left(\sec^2B\right)\left(C+2D\cos2B\right)+D^2\left(1+\tan^2B\right)}+\\&+\frac{\sin\left(\nu_A^B+180^\circ-\alpha_1+A-B\right)\cdot\sin\alpha_2\cdot\sin\beta_1\cdot\sin\beta_2\cdot\sin\left(\alpha_2+2B\right)\cdot\sin2B\cdot2\cos\left(B-A\right)\cdot\tan B}{C\left(\sec^2B\right)\left(C+2D\cos2B\right)+D^2\left(1+\tan^2B\right)}\Big\}\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial Y_1}{\partial\alpha_2}&=2d\Bigg[\frac{\sin\beta_1\cdot\sin\beta_2\cdot\sin2B\cdot\sin\left(\alpha_1+2B_1\right)\cdot\tan B}{C\left(\sec^2B\right)\left(C+2D\cos2B\right)+D^2\left(1+\tan^2B\right)}\cdot\\&\cdot\frac{\sin\left(B-A\right)\cdot\cos\left(\nu_A^B+180^\circ-\alpha_1+A-B\right)-\cos\left(B-A\right)\cdot\sin\left(\nu_A^B+180^\circ-\alpha_1+A-B\right)}{C\left(\sec^2B\right)\left(C+2D\cos2B\right)+D^2\left(1+\tan^2B\right)}\Bigg]\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\partial Y_1}{\partial\beta_1}&=-4d\Bigg[\frac{\sin\left(\alpha_1-\alpha_2\right)\cdot\sin\beta_2\cdot\sin\left(\alpha_2+\beta_2\right)\cdot\sin^2B}{C\cdot\left(\sec^2B\right)\left(C+2D\cos2B\right)+D^2\left(1+\tan^2B\right)}\cdot\\&\cdot\frac{\sin\left(B-A\right)\cdot\cos\left(\nu_A^B+180^\circ-\alpha_1+A-B\right)-\cos\left(B-A\right)\cdot\sin\left(\nu_A^B+180^\circ-\alpha_1+A-B\right)}{C\cdot\left(\sec^2B\right)\left(C+2D\cos2B\right)+D^2\left(1+\tan^2B\right)}\Bigg]\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{\partial Y_1}{\partial\beta_2}&=d\Bigg\{\frac{\sin^2B\left[-2\sin\alpha_1+\sin\left(\alpha_1+2\beta_2\right)+\sin\left(\alpha_1-2\alpha_2-2\beta_2\right)-\sin\left(\alpha_1+4B\right)+\sin\left(\alpha_1+2\alpha_2+4B\right)\right]}{C\left(\sec^2B\right)\left(C+2D\cos2B\right)+D^2\left(1+\tan^2B\right)}\cdot\\&\cdot\frac{\sin\left(B-A\right)\cdot\cos\left(\nu_A^B+180^\circ-\alpha_1+A-B\right)-\cos\left(B-A\right)\cdot\sin\left(\nu_A^B+180^\circ-\alpha_1+A-B\right)}{C\left(\sec^2B\right)\left(C+2D\cos2B\right)+D^2\left(1+\tan^2B\right)}\Bigg\}\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\partial X_1}{\partial\alpha_1}&=d\cdot\csc\alpha_1\Bigg\{-\cos\left(\nu_A^B+180^\circ-\alpha_1+A-B\right)\cdot\sin\left(B-A\right)\cdot\cot\alpha_1+\frac{\sin\left(\alpha_2+\beta_2\right)\cdot\sin^2B\cdot\sin\left(B-A\right)}{C\left(\sec^2B\right)\left(C+2D\cos2B\right)+D^2\left(1+\tan^2B\right)}\cdot\\&+\frac{\left[\sin\left(2\alpha_1-\alpha_2-\beta_2\right)-\sin\left(2\alpha_1-\alpha_2+\beta_2\right)+2\sin\left(\alpha_2+\beta_2\right)+\sin\left(2\alpha_1-\alpha_2+2\beta_1+\beta_2\right)-\sin\left(2\alpha_1+\alpha_2+2\beta_1+\beta_2\right)\right]}{C\left(\sec^2B\right)\left(C+2D\cos2B\right)+D^2\left(1+\tan^2B\right)}+\\&+\frac{2\cos\left(B-A\right)\cdot\cos\left(\nu_A^B+180^\circ-\alpha_1+A-B\right)\cdot\sin\alpha_2\cdot\sin\beta_1\cdot\sin\beta_2\cdot\sin\left(\alpha_2+2B\right)\cdot\sin2B\cdot\tan B}{C\left(\sec^2B\right)\left(C+2D\cos2B\right)+D^2\left(1+\tan^2B\right)}\Bigg\}\end{aligned} \tag{29}$$

$$\frac{\partial X_1}{\partial \alpha_2} = -2d\left[\frac{\sin(\alpha_1+2B)\cdot\sin\beta_1\cdot\sin\beta_2\cdot\sin 2B\cdot\tan B}{C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)}\cdot\right.$$
$$\left.\cdot\frac{\cos(B-A)\cdot\cos(\nu_A^B+180^\circ-\alpha_1+A-B)-\sin(B-A)\cdot\sin(\nu_A^B+180^\circ-\alpha_1+A-B)}{C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)}\right] \quad (30)$$

$$\frac{\partial X_1}{\partial \beta_1} = 4d\left[\frac{\sin(\alpha_1-\alpha_2)\cdot\sin(\alpha_2+\beta_2)\cdot\sin\beta_2\cdot\sin^2 B}{C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)}\cdot\right.$$
$$\left.\cdot\frac{\cos(B-A)\cdot\cos(\nu_A^B+180^\circ-\alpha_1+A-B)+\sin(B-A)\cdot\sin(\nu_A^B+180^\circ-\alpha_1+A-B)}{C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)}\right] \quad (31)$$

$$\frac{\partial X_1}{\partial \beta_2} = -d\left\{\frac{\sin^2 B\left[-2\sin\alpha_1+\sin(\alpha_1+2\beta_2)+\sin(\alpha_1-2\alpha_2-2\beta_2)-\sin(\alpha_1+4B)+\sin(\alpha_1+2\alpha_2+4B)\right]}{C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)}\cdot\right. \quad (32)$$
$$\left.\cdot\frac{\cos(B-A)\cdot\cos(\nu_A^B+180^\circ-\alpha_1+A-B)-\sin(B-A)\cdot\sin(\nu_A^B+180^\circ-\alpha_1+A-B)}{C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)}\right\}$$

$$\frac{\partial Y_2}{\partial \alpha_1} = 2d\left[\frac{\sin\beta_1\cdot\sin\beta_2\cdot\sin 2B\cdot\sin(\alpha_2+2B)\cdot\sin(180^\circ-\nu_B^A-\alpha_2-2A-2B)\cdot\tan B}{C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)}\right] \quad (33)$$

$$\frac{\partial Y_2}{\partial \alpha_2} = d\csc\alpha_2\left\{\sin(180^\circ-\nu_B^A-\alpha_2-A-B)\cdot\cot\alpha_2\cdot\sin(A+B)-\right.$$
$$-\frac{\left[-\sin(\alpha_1-2\alpha_2-\beta_1)-2\sin(\alpha_1+\beta_1)+\sin(\alpha_1-2\alpha_2+\beta_1)+\sin(\alpha_1-2\alpha_2-\beta_1-2\beta_2)+\sin(\alpha_1+2\alpha_2+\beta_1+2\beta_2)\right]}{C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)}\cdot$$
$$\cdot\frac{\sin(\alpha_1+\beta_1)\cdot\sin^2 B\cdot\cos(180^\circ-\nu_B^A-\alpha_2-A-B)\cdot\sin(A+B)}{C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)}-$$
$$\left.-\frac{\sin(180^\circ-\nu_B^A-\alpha_2-A-B)\cdot 2\cos(A+B)\cdot\sin\alpha_1\cdot\sin\beta_1\cdot\sin\beta_2\cdot\sin(\alpha_1+2B)\cdot\sin 2B\cdot\tan B}{C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)}\right\}$$
$$(34)$$

$$\frac{\partial Y_2}{\partial \beta_1} = d\left\{\frac{\left[-2+2\cos(2\alpha_2)+\cos(2\beta_1)+\cos(2\alpha_1+2\beta_1)-\cos(2\alpha_1-2\alpha_2+2\beta_1)-\cos(2\alpha_2+2\beta_1)-\cos 4B+\right.}{2\left[C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)\right]}\right. \quad (35)$$
$$\left.\frac{\left.+\cos(2\alpha_1+4B)+\cos(2\alpha_2+4B)-\cos(2\alpha_1+2\alpha_2+4B)\right]\cdot\csc^2\alpha_2\cdot\sin^2 B\cdot\sin(180^\circ-\nu_B^A-\alpha_2-2A-2B)}{2\left[C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)\right]}\right\}$$

$$\frac{\partial Y_2}{\partial \beta_2} = 4d\left[\frac{\sin(\alpha_1-\alpha_2)\cdot\sin(\beta_1)\cdot\sin(\alpha_1+\beta_1)\cdot\sin^2 B\cdot\sin(180^\circ-\nu_B^A-\alpha_2-2A-2B)}{C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)}\right] \quad (36)$$

$$\frac{\partial X_2}{\partial \alpha_1} = -2d\left[\frac{\sin\beta_1\cdot\sin\beta_2\cdot\sin 2B\cdot\sin(\alpha_2+2B)\cdot\tan B\cdot\cos(180^\circ-\nu_B^A-\alpha_2-2A-2B)}{C(\sec^2 B)(C+2D\cos 2B)+D^2(1+\tan^2 B)}\right] \quad (37)$$

$$\frac{\partial X_2}{\partial \alpha_2} = d\csc\alpha_2 \left\{ -\cos\left(180^\circ - \nu_B^A - \alpha_2 - A - B\right)\cdot\cot\alpha_2\cdot\sin\left(A+B\right) - \right.$$
$$-\frac{\left[-\sin\left(\alpha_1 - 2\alpha_2 - \beta_1\right) - 2\sin\left(\alpha_1+\beta_1\right) + \sin\left(\alpha_1 - 2\alpha_2 + \beta_1\right) + \sin\left(\alpha_1 - 2\alpha_2 - \beta_1 - 2\beta_2\right) + \sin\left(\alpha_1 + 2\alpha_2 + \beta_1 + 2\beta_2\right)\right]}{C\left(\sec^2 B\right)\left(C + 2D\cos 2B\right) + D^2\left(1+\tan^2 B\right)}\cdot$$
$$\cdot\frac{\sin\left(\alpha_1+\beta_1\right)\cdot\sin^2 B\cdot\sin\left(180^\circ - \nu_B^A - \alpha_2 - A - B\right)\cdot\sin\left(A+B\right)}{C\left(\sec^2 B\right)\left(C+2D\cos 2B\right) + D^2\left(1+\tan^2 B\right)} +$$
$$\left. + \frac{2\cos\left(180^\circ - \nu_B^A - \alpha_2 - A - B\right)\cdot\cos\left(A+B\right)\cdot\sin\alpha_1\cdot\sin\beta_1\cdot\sin\beta_2\cdot\sin 2B\cdot\sin\left(\alpha_1+2B\right)\cdot\tan B}{C\left(\sec^2 B\right)\left(C+2D\cos 2B\right) + D^2\left(1+\tan^2 B\right)} \right\} \tag{38}$$

$$\frac{\partial X_2}{\partial \beta_1} = d\left\{ \frac{\left[2 - 2\cos\left(2\alpha_2\right) - \cos\left(2\beta_1\right) - \cos\left(2\alpha_1 + 2\beta_1\right) + \cos\left(2\alpha_1 - 2\alpha_2 + 2\beta_1\right) + \cos\left(2\alpha_2 + 2\beta_1\right) + \cos 4B - \right.}{2\left[C\left(\sec^2 B\right)\left(C+2D\cos 2B\right) + D^2\left(1+\tan^2 B\right)\right]} \right. \tag{39}$$
$$\left. \frac{\left. -\cos\left(2\alpha_1 + 4B\right) - \cos\left(2\alpha_2 + 4B\right) + \cos\left(2\alpha_1 + 2\alpha_2 + 4B\right)\right]\cdot\csc^2\alpha_2\cdot\sin^2 B\cdot\cos\left(180^\circ - \nu_B^A - \alpha_2 - 2A - 2B\right)}{2\left[C\left(\sec^2 B\right)\left(C+2D\cos 2B\right) + D^2\left(1+\tan^2 B\right)\right]} \right\}$$

$$\frac{\partial X_2}{\partial \beta_2} = -4d\left[\frac{\sin\left(\alpha_1 - \alpha_2\right)\cdot\sin\left(\beta_1\right)\cdot\sin\left(\alpha_1+\beta_1\right)\cdot\sin^2 B\cdot\cos\left(180^\circ - \nu_B^A - \alpha_2 - 2A - 2B\right)}{C\left(\sec^2 B\right)\left(C+2D\cos 2B\right) + D^2\left(1+\tan^2 B\right)}\right] \tag{40}$$

Where shifts are:
$$C = \sin\alpha_2\cdot\sin\beta_1\cdot\sin\left(\alpha_1+\beta_1+\beta_2\right)$$
$$D = \sin\alpha_1\cdot\sin\beta_2\cdot\sin\left(\alpha_2+\beta_1+\beta_2\right)$$

3.1. Example

An example of the analysis pertaining to the standard deviation of unknown points has been performed on the geodetic quadrilateral in the shape of a regular square whose elements are shown in Figure 2.

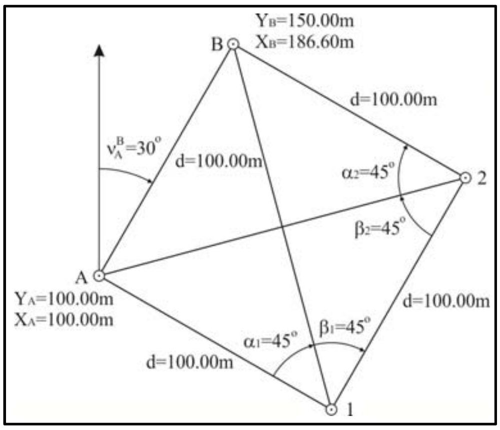

Figure 2 - Geodetic quadrilateral in the shape of a regular square

The coordinates of unknown points *1* and *2* have been evaluated using equations (13) and (15):

$$Y_1 = 186.60\text{m}; \quad X_1 = 50.00\text{m}$$

$$Y_2 = 236.60\text{m}; \quad X_2 = 136.60\text{m}$$

The values of the partial derivatives are:

$$\frac{\partial Y_1}{\partial \alpha_1} = +100\text{m}; \quad \frac{\partial Y_1}{\partial \alpha_2} = -136.603\text{m}; \quad \frac{\partial Y_1}{\partial \beta_1} = 0; \quad \frac{\partial Y_1}{\partial \beta_2} = +136.603\text{m}$$

$$\frac{\partial X_1}{\partial \alpha_1} = +173.206\text{m}; \quad \frac{\partial X_1}{\partial \alpha_2} = -36.603\text{m}; \quad \frac{\partial X_1}{\partial \beta_1} = 0; \quad \frac{\partial X_1}{\partial \beta_2} = +36.603\text{m}$$

$$\frac{\partial Y_2}{\partial \alpha_1} = -36.603\text{m}; \quad \frac{\partial Y_2}{\partial \alpha_2} = -100\text{m}; \quad \frac{\partial Y_2}{\partial \beta_1} = +36,603\text{m}; \quad \frac{\partial Y_2}{\partial \beta_2} = 0$$

$$\frac{\partial X_2}{\partial \alpha_1} = +136.603\text{m}; \quad \frac{\partial X_2}{\partial \alpha_2} = -173.206\text{m}; \quad \frac{\partial X_2}{\partial \beta_1} = -136.603\text{m}; \quad \frac{\partial X_2}{\partial \beta_2} = 0$$

Based on these values and assuming various standard deviations of the measured angles using the equations (21-24), the standard deviations of the coordinates of the points *1* and *2* have been evaluated, where:

$$\sigma_1 = \sqrt{\sigma_{Y_1}{}^2 + \sigma_{X_1}{}^2}; \quad \sigma_2 = \sqrt{\sigma_{Y_2}{}^2 + \sigma_{X_2}{}^2} \tag{41}$$

Table 1- Standard deviation of the coordinates of unknown points

	$\sigma_{\alpha_1} = \sigma_{\alpha_2} = \sigma_{\beta_1} = \sigma_{\beta_2}$							
	1"	**2"**	**3"**	**5"**	**10"**	**20"**	**30"**	**60"**
σ_{Y_1} **[mm]**	1.1	2.1	3.2	5.3	10.5	21.1	31.6	63.3
σ_{X_1} **[mm]**	0.9	1.8	2.6	4.4	8.8	17.5	26.3	52.6
σ_1 **[mm]**	1.4	2.7	4.1	6.9	13.7	27.4	41.1	82.3
σ_{Y_2} **[mm]**	0.5	1.1	1.6	2.7	5.5	10.9	16.4	32.8
σ_{X_2} **[mm]**	1.3	2.5	3.8	6.3	12.6	25.2	37.7	75.5
σ_2 **[mm]**	1.4	2.7	4.1	6.9	13.7	27.4	41.1	82.3

As Table 1 shows, considering the regular shape of the geodetic quadrilateral, the standard deviations of the unknown points are mutually equal, and their value increases proportional to the decrease in the accuracy of the angle measurements in the quadrilateral.

4. CONCLUSION

The standard deviation of the coordinates of unknown points which is determined using Hansen's method is not presented in scientific literature. One reason for this, among others, must lie in the fact that it involves extensive and complex calculations, primarily of the coefficients which represent partial derivatives with respect to variable or measured quantities. Although contemporary measuring techniques enable the determining of the coordinates of any point providing there is mutual visibility, there is a whole range of different situations on the terrain which require the application of Hansen's method. This is, above all, the case when solving specific engineering problems in regard to mines with underground exploitation.

For this reason, it is vital to perceive the optimal shape of the geodetic quadrilateral, having in mind that errors in point coordinates, besides being influenced by errors in the measured angles, are also impacted by the shape of the quadrilateral. This paper serves as the first step in the analysis of the impact of the geodetic quadrilateral shape on the standard deviation of the coordinates of the unknown points where standard deviation equations have been derived.

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